

THE DUAL OF A NON-REFLEXIVE L-EMBEDDED BANACH SPACE CONTAINS l^∞ ISOMETRICALLY

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ABSTRACT. See title. (A Banach space is said to be L-embedded if it is complemented in its bidual such that the norm between the two complementary subspaces is additive.)

This note is an afterthought to a result of Dowling [2] according to which a dual Banach space contains an isometric copy of c_0 if it contains an asymptotic one. (For definitions see below.) It is known ([7] or [4, Th. IV.2.7]) that the dual of a non-reflexive L-embedded Banach space contains c_0 isomorphically. For a special class of L-embedded Banach spaces the construction of the c_0 -copy has been improved so to yield an asymptotic one ([8, Prop. 6]) and it turns out that this improvement is possible in the general case which together with Dowling's result yields isometric copies of c_0 in the dual of an L-embedded Banach space.

All this is perhaps known - or at least not surprising - to experts in the field but the final result, i.e. an isometric copy of c_0 (and hence of l^∞) in the dual of an L-embedded space X , is optimal in the category of Banach spaces and therefore it seems worthwhile proving it explicitly. As in [7] we will prove a bit more by constructing the c_0 -copy within the context of Pełczyński's property (V^*) i.e. the c_0 -basis will be constructed so to behave approximately like biorthogonal functionals on the basis of a given l^1 -basis in X , see (2) and (3) below where in particular the value $\tilde{c}_J(x_n)$ in (2) is optimal. (For the definition and some basic results on Pełczyński's property (V^*) see [4].)

Preliminaries: A projection P on a Banach space Z is called an *L-projection* if $\|Pz\| + \|z - Pz\| = \|z\|$ for all $z \in Z$. A Banach space X is called *L-embedded* (or *an L-summand in its bidual*) if it is the image of an L-projection on its bidual. In this case we write $X^{**} = X \oplus_1 X_s$. Among classical Banach spaces, the Hardy space H_0^1 , L^1 -spaces and, more generally, the preduals of von Neumann algebras or of JBW*-triples serve as examples of L-embedded spaces. A sequence (x_n) in a Banach space X is said to *span c_0 asymptotically isometrically* (or *just to span c_0 asymptotically*) if there is a null sequence (δ_n) in $[0, 1[$ such that $\sup(1 - \delta_n)|\alpha_n| \leq \left\| \sum \alpha_n x_n \right\| \leq \sup(1 + \delta_n)|\alpha_n|$ for all $(\alpha_n) \in c_0$. X is said to contain c_0 asymptotically if it contains such a sequence (x_n) . Recall the routine fact that if (x_n^*) in X^* is equivalent to the canonical basis of c_0 then $\sum \alpha_n x_n^*$ makes sense for all $(\alpha_n) \in l^\infty$ in the w^* -topology of X^* and by lower w^* -semicontinuity of the norm an estimate $\left\| \sum \alpha_n x_n^* \right\| \leq M \sup |\alpha_n|$ that holds for all $(\alpha_n) \in c_0$

1991 *Mathematics Subject Classification*. Primary 46B20; Secondary 46B03, 46B04, 46B26.

Key words and phrases. L-embedded Banach spaces, isometric copies of c_0 .

extends to all $(\alpha_n) \in l^\infty$. The Banach spaces we consider in this note are real or complex, the set \mathbb{N} starts at 1.

To a bounded sequence (x_n) in a Banach space X we associate its 'James constant'

$$c_J(x_n) = \sup c_m \quad \text{where the} \quad c_m = \inf_{\sum_{n \geq m} |\alpha_n| = 1} \left\| \sum_{n \geq m} \alpha_n x_n \right\|$$

form an increasing sequence. If (x_n) is equivalent to the canonical basis of l^1 then $c_J(x_n) > 0$ and more specifically, $c_J(x_n) > 0$ if and only if there is an integer m such that $(x_n)_{n \geq m}$ is equivalent to the canonical basis of l^1 . Roughly speaking, the number $c_J(x_n)$ may be thought of as the 'approximately best l^1 -basis constant' of (x_n) ; precisely speaking, there is a null sequence (τ_m) in $[0, 1[$ (determined by $c_m = (1 - \tau_m)c_J(x_n)$) such that $\left\| \sum_{n=m}^\infty \alpha_n x_n \right\| \geq (1 - \tau_m)c_J(x_n) \sum_{n=m}^\infty |\alpha_n|$ for all $(\alpha_n) \in l^1$ and $c_J(x_n)$ cannot be replaced by a strictly greater constant. If one passes to a subsequence (x_{n_k}) of (x_n) then $c_J(x_{n_k}) \geq c_J(x_n)$ hence it makes sense to define

$$\tilde{c}_J(x_n) = \sup_{n_k} c_J(x_{n_k}).$$

The standard reference for L-embedded Banach spaces is the monograph [4, Ch. IV]. For general Banach space theory and undefined notation we refer to [1], [5], or [6].

The main result of this note is

Theorem 1. *Let X be an L-embedded Banach space and let (x_n) be equivalent to the canonical basis of l^1 . Then there is a sequence (x_n^*) in X^* that generates l^∞ isometrically, more precisely*

$$(1) \quad \left\| \sum \alpha_n x_n^* \right\| = \sup |\alpha_n| \text{ for all } (\alpha_n) \in l^\infty$$

and there is a strictly increasing sequence (p_n) in \mathbb{N} such that

$$(2) \quad \lim |x_n^*(x_{p_n})| = \tilde{c}_J(x_m)$$

$$(3) \quad x_n^*(x_{p_l}) = 0 \quad \text{if } l < n$$

In particular, the dual of a non-reflexive L-embedded Banach space contains an isometric copy of l^∞ .

In order to prove the theorem we first state and prove Dowling's result in a way which fits our purpose.

Proposition 2. *Let (ε_n) be a null sequence in $[0, 1[$, let (N_n) be a sequence of pairwise disjoint infinite subsets of \mathbb{N} and let (y_n^*) in the dual of a Banach space Y span c_0 such that*

$$(4) \quad \left\| \sum \alpha_n y_n^* \right\| \leq \sup (1 + \varepsilon_n) |\alpha_n| \quad \text{and} \quad \|y_n^*\| \rightarrow 1$$

for all $(\alpha_n) \in c_0$.

Then the elements

$$(5) \quad x_n^* = \sum_{k \in N_n} \frac{y_k^*}{1 + \varepsilon_k}$$

generate l^∞ isometrically (as in (1)).

Proof of the proposition: With (ε_n) , (N_n) and (y_n^*) as in the hypothesis of the statement define x_n^* by (5). Then $\|x_n^*\| \leq 1$ for all $n \in \mathbb{N}$ by the first half of (4). For the inverse inequality we have that

$$\|x_n^*\| \geq \left\| 2 \frac{y_m^*}{1 + \varepsilon_m} - \left\| \frac{y_m^*}{1 + \varepsilon_m} - \sum_{k \in N_n, k \neq m} \frac{y_k^*}{1 + \varepsilon_k} \right\| \right\| \geq 2 \frac{\|y_m^*\|}{1 + \varepsilon_m} - 1$$

holds for all $m \in N_n$ hence $\|x_n^*\| \geq 1$ by the second half of (4) which proves $\|x_n^*\| = 1$.

Similarly we show (1): First, “ \leq ” of (1) follows from the first half of (4); second, by the just shown inequality we have

$$\left\| \sum \alpha_n x_n^* \right\| \geq 2|\alpha_m| - \|\alpha_m x_m^* - \sum_{n \neq m} \alpha_n x_n^*\| \geq 2|\alpha_m| - \sup |\alpha_n|$$

for all $m \in \mathbb{N}$ hence “ \geq ” of (1). ■

Proof of the theorem:

Let (δ_n) be a sequence in $]0, 1[$ converging to 0. Suppose (x_n) is an l^1 -basis and write $\tilde{c} = \tilde{c}_J(x_n)$ for short.

Observation: Given $\tau > 0$ there is a subsequence (x_{n_k}) of (x_n) such that $c_J(x_{n_k}) > (1 - \tau)\tilde{c}$ and by James’ l^1 -distortion theorem there are blocks of the x_{n_k} which span l^1 almost isometrically that is to say there are pairwise disjoint finite sets $A_l \subset \{n_k \mid k \in \mathbb{N}\}$, a sequence of scalars (λ_n) such that $\sum_{k \in A_l} |\lambda_k| = 1$ and such that the sequence (z_l) defined by $z_l = \tilde{z}_l / \|\tilde{z}_l\|$ and $\tilde{z}_l = \sum_{k \in A_l} \lambda_k x_k$ satisfies $(1 - 2^{-m}) \sum_{l=m}^\infty |\alpha_l| \leq \left\| \sum_{l=m}^\infty \alpha_l z_l \right\| \leq (1 + 2^{-m}) \sum_{l=m}^\infty |\alpha_l|$ for all $m \in \mathbb{N}$; furthermore $\|\tilde{z}_l\| \rightarrow c_J(x_{n_k})$ whence the existence of l' such that $|\tilde{c} - \|\tilde{z}_{l'}\|| < \tau$.

By induction over $n \in \mathbb{N}$ we will construct finite sequences $(y_i^{(n)*})_{i=1}^n$ in X^* , a sequence (\tilde{y}_n) in X , pairwise disjoint finite sets $C_n \subset \mathbb{N}$ and a scalar sequence (μ_n) such that, with the notation $y_n = \tilde{y}_n / \|\tilde{y}_n\|$,

$$(6) \quad \sum_{k \in C_n} |\mu_k| = 1, \quad \tilde{y}_n = \sum_{k \in C_n} \mu_k x_k, \quad |\tilde{c} - \|\tilde{y}_n\|| < \delta_n,$$

$$(7) \quad |y_i^{(n)*}(y_i)| > 1 - \delta_i \quad \forall i \leq n,$$

$$(8) \quad y_i^{(n)*}(y_l) = 0 \quad \forall l < i \leq n,$$

$$(9) \quad y_i^{(n)*}(x_p) = 0 \quad \forall p \in C_l, \forall l < i \leq n,$$

$$(10) \quad \left\| \sum_{i=1}^m \alpha_i y_i^{(n)*} \right\| \leq \max_{i \leq m} (1 + (1 - 2^{-n}) \delta_i) |\alpha_i| \quad \forall m \leq n, \alpha_i \text{ scalars.}$$

For $n = 1$ we use the observation with $\tau = \delta_1$ and choose l_1 such that $|\|\tilde{z}_{l_1}\| - \tilde{c}| < \delta_1$. Then we choose $y_1^{(1)*}$ such that $\|y_1^{(1)*}\| = 1$ and $y_1^{(1)*}(z_{l_1}) = \|\tilde{z}_{l_1}\|$. It remains to set $C_1 = A_{l_1}$, $\mu_k = \lambda_k$ for $k \in C_1$ and $\tilde{y}_1 = \tilde{z}_{l_1}$.

For the induction step $n \mapsto n + 1$ we recall that $(P^*)|_{X^*}$ is an isometric isomorphism from X^* onto X_s^\perp , that $X^{***} = X^\perp \oplus_\infty X_s^\perp$ and that $(P^* x^*)|_X = (x^*)|_X$ for all $x^* \in X^*$. Let (z_l) be as in the observation above with $\tau = \delta_{n+1}$ and let $z_s \in X^{**} \setminus X$ be a w^* -accumulation point of the z_l . Then $z_s \in X_s$ and $\|z_s\| = 1$ by the proof of [8, Lem 1] (or by some general folklore argument). Choose $t \in \ker P^* \subset X^{***}$ such that $\|t\| = 1$

and $t(z_s) = \|z_s\|$. Put

$$\begin{aligned} E &= \text{lin}(\{P^*y_i^{(m)*} \mid i \leq m \leq n\} \cup \{t\}) \subset X^{***}, \\ F &= \text{lin}(\{y_i \mid i \leq n\} \cup \{z_s\} \cup \{x_p \mid p \in \bigcup_{l \leq n} C_l\}) \subset X^{**} \end{aligned}$$

and choose $\eta > 0$ such that

$$(1 + \eta)(1 + (1 - 2^{-n})\delta_i) < 1 + (1 - 2^{-(n+1)})\delta_i \text{ and } \eta < (1 - 2^{-(n+1)})\delta_{n+1}$$

for all $i \leq n$. The principle of local reflexivity provides an operator $R : E \rightarrow X^*$ such that

$$(11) \quad (1 - \eta)\|e^{***}\| \leq \|Re^{***}\| \leq (1 + \eta)\|e^{***}\|,$$

$$(12) \quad f^{**}(Re^{***}) = e^{***}(f^{**}),$$

for all $e^{***} \in E$ and $f^{**} \in F$.

We define $y_i^{(n+1)*} = R(P^*y_i^{(n)*})$ for $i \leq n$ and $y_{n+1}^{(n+1)*} = Rt$ and obtain (10, $n + 1$) (with $\alpha_i = 0$ if $m < i \leq n + 1$) by

$$\begin{aligned} \left\| \sum_{i=1}^{n+1} \alpha_i y_i^{(n+1)*} \right\| &\stackrel{(11)}{\leq} (1 + \eta) \left\| \left(\sum_{i=1}^n \alpha_i P^* y_i^{(n)*} \right) + \alpha_{n+1} t \right\| \\ &= (1 + \eta) \max \left(\left\| \sum_{i=1}^n \alpha_i P^* y_i^{(n)*} \right\|, \|\alpha_{n+1} t\| \right) \\ &= (1 + \eta) \max \left(\left\| \sum_{i=1}^n \alpha_i y_i^{(n)*} \right\|, \|\alpha_{n+1} t\| \right) \\ &\stackrel{(10)}{\leq} (1 + \eta) \max \left(\max_{i \leq n} (1 + (1 - 2^{-n})\delta_i) |\alpha_i|, |\alpha_{n+1}| \right) \\ &\leq \max_{i \leq n+1} (1 + (1 - 2^{-(n+1)})\delta_i) |\alpha_i|. \end{aligned}$$

Since z_s is a w^* -cluster point of (z_l) we have

$$\begin{aligned} |y_{n+1}^{(n+1)*}(z_l)| &> |z_s(y_{n+1}^{(n+1)*})| - \delta_{n+1} \\ &\stackrel{(12)}{=} |t(z_s)| - \delta_{n+1} = 1 - \delta_{n+1} \end{aligned}$$

for infinitely many l ; furthermore, an l_{n+1} can be chosen among these l so to obtain $\|\tilde{z}_{l_{n+1}}\| - \tilde{c} < \delta_{n+1}$. Set $C_{n+1} = A_{l_{n+1}}$, $\tilde{y}_{n+1} = \tilde{z}_{l_{n+1}}$, $\mu_k = \lambda_k$ for $k \in C_{n+1}$. Then (6) holds and (7, $n + 1$) holds for $i = n + 1$. For $i \leq n$, (7, $n + 1$) follows from

$$y_i^{(n+1)*}(y_i) = (P^*y_i^{(n)*})(y_i) = y_i^{(n)*}(y_i) \stackrel{(7)}{>} 1 - \delta_i.$$

Condition (8, $n + 1$) holds for $i = n + 1$ by

$$y_{n+1}^{(n+1)*}(y_l) = (Rt)(y_l) \stackrel{(12)}{=} t(y_l) = 0 \quad \forall l < n + 1$$

and it holds for $i < n + 1$ by

$$y_i^{(n+1)*}(y_l) = (P^*y_i^{(n)*})(y_l) = y_i^{(n)*}(y_l) \stackrel{(8)}{=} 0 \quad \forall l < i.$$

The proof of (9, $n + 1$) works like the one of (8, $n + 1$). This ends the induction.

Now we define $y_i^* = \frac{1}{1+\delta_i} \lim_{n \in \mathcal{U}} y_i^{(n)*}$ for all $i \in \mathbb{N}$ where \mathcal{U} is a fixed nontrivial ultrafilter on \mathbb{N} and where the limit is understood in the w^* -topology of X^* . Then by w^* -lower semicontinuity of the norm and by (10)

$$\left\| \sum \alpha_i y_i^* \right\| \leq \sup(1 + \delta_i) \frac{|\alpha_i|}{1 + \delta_i} = \sup |\alpha_i|$$

for all $(\alpha_i) \in l^\infty$. In particular, $\|y_i^*\| \leq 1$ hence $\|y_i^*\| \rightarrow 1$ by (7) and (y_i^*) satisfies (4) for $\varepsilon_n = 0$.

Let (N_n) be a sequence of pairwise disjoint infinite subsets of \mathbb{N} such that (i_n) increases strictly where $i_n = \min N_n$. By the proposition the sequence defined by

$$x_n^* = \sum_{i \in N_n} y_i^*$$

generates l^∞ isometrically and we have $|x_n^*(y_{i_n})| \stackrel{(8)}{=} |y_{i_n}^*(y_{i_n})| \stackrel{(7)}{\geq} 1 - \delta_{i_n}$. By construction of the y_i there is, for each $n \in \mathbb{N}$, an index $p_n \in C_{i_n}$

such that $|x_n^*(x_{p_n})| \geq (1 - \delta_{i_n}) \|\tilde{y}_{i_n}\| \stackrel{(6)}{\geq} (1 - \delta_{i_n})(\tilde{c} - \delta_{i_n})$ which will yield “ \geq ” of (2). In order to show “ \leq ” of (2) suppose to the contrary that $x_{n_m}^*(x_{p_{n_m}}) > \kappa + \tilde{c}$ for appropriate subsequences, all m and $\kappa > 0$. According to an extraction lemma of Simons [10] we may furthermore suppose that $\sum_{j \neq m} |x_{n_j}^*(x_{p_{n_m}})| < \kappa/2$ for all m . Then given (α_m) and θ_m such that $\theta_m \alpha_m = |\alpha_m|$ we obtain $\left\| \sum \alpha_m x_{p_{n_m}} \right\| \geq (\sum_j \theta_j x_{n_j}^*)(\sum_m \alpha_m x_{p_{n_m}}) \geq (\kappa + \tilde{c}) \sum_m |\alpha_m| - \sum_m \sum_{j \neq m} |\alpha_m| |x_{n_j}^*(x_{p_{n_m}})| \geq ((\kappa/2) + \tilde{c}) \sum_m |\alpha_m|$ which yields the contradiction $c_J(x_{p_{n_m}}) > \tilde{c}$ and thus shows “ \leq ” and all of (2) whereas (3) follows from (9) via $y_i^*(x_p) = 0$ for $p \in C_l$, $l < i$.

The last assertion of the theorem is immediate from the fact that non-reflexive L-embedded spaces contain l^1 isomorphically [4, IV.2.3] ■

Remarks:

1. It is not clear whether (3) can be obtained also for $l > n$. What can be said by Simons’ extraction lemma (used in the proof) is that, under the assumptions of the theorem and given $\varepsilon > 0$, it is possible (after passing to appropriate subsequences) to obtain in addition to (3) that $\sum_{n=1}^{l-1} |x_n^*(x_{p_l})| = \sum_{n \neq l} |x_n^*(x_{p_l})| < \varepsilon$ for all l . In case $\tilde{c}_J(x_n) = 1 = \lim \|x_n\|$ (which happens when the x_n span l^1 almost isometrically) this can be improved to

$$(13) \sum_{n \neq l} |x_n^*(x_{p_l})| = \left(\sum_n |x_n^*(x_{p_l})| \right) - |x_l^*(x_{p_l})| \leq \|x_{p_l}\| - |x_l^*(x_{p_l})| \rightarrow 0.$$

One might also construct straightforward perturbations of the x_n^* in order to get (3) for $l \neq n$ but then it is not clear whether these perturbations can be arranged to span c_0 isometrically, not just almost isometrically.

Since in general L-embedded spaces do not contain l^1 isometrically (see below, last remark) it is in general not possible to improve (2) and (3) so to obtain $x_n^*(x_{p_l}) = \tilde{c}(x_m)$ if $l = n$ and $= 0$ if $l \neq n$.

2. As already alluded to in the introduction, the construction of c_0 in this paper bears much resemblance to the one of [7]. A different way to construct c_0 is contained in [9] but it seems unlikely that this construction can be improved to yield an isometric c_0 -copy.

3. It follows from (2) that in L-embedded spaces the sup in the definition of \tilde{c}_J is attained by the James constant of an appropriate subsequence. For

general Banach spaces this is not known although it can be shown by a routine diagonal argument that each bounded sequence (x_n) admits a c_J -stable subsequence (x_{n_k}) (meaning that $\tilde{c}_J(x_{n_k}) = c_J(x_{n_k})$) whose James constant is arbitrarily near to $\tilde{c}(x_n)$.

4. Each normalized sequence (x_n) in an L-embedded Banach space that spans l^1 almost isomorphically contains a subsequence each of whose w^* -accumulation points in the bidual attains its norm on the dual unit ball. To see this let (x_n^*) and (x_{p_n}) be the sequences given by the theorem and by Simons' extraction lemma (see (13) above), let x_s be a w^* -accumulation point of the x_{p_n} and let $x^* = \sum x_n^*$; then $\|x^*\| = 1$ and on the one hand $\|x_s\| = 1$ by [8] and on the other hand $x_s(x^*) = \lim x^*(x_{p_n}) \stackrel{(13)}{=} \lim x_n^*(x_{p_n}) \stackrel{(2)}{=} c_J(x_n) = 1$.

It would be interesting to know whether this remark holds for the whole sequence (x_n) instead of only a subsequence (x_{p_n}) . A kind of converse follows from [9, Rem. 2] for separable X : If $x_s \in X_s$ attains its norm on the dual unit ball then it does so on the sum of a wuC-series.

5. Let us finally note that the presence of isometric c_0 -copies in X^* does not necessarily entail the presence of isometric copies of l^1 in X even if X is the dual of an M-embedded Banach space. This follows from [4, Cor. III.2.12] which states that there is an L-embedded Banach space which is the dual of an M-embedded space (to wit the dual of c_0 with an equivalent norm) which is strictly convex and therefore does not contain l^1 isometrically although it contains, as do all non-reflexive L-embedded spaces, l^1 asymptotically ([8], see [3] for the definition of asymptotic copies and the difference to almost isometric ones).

REFERENCES

- [1] J. Diestel. *Sequences and Series in Banach Spaces*. Springer, Berlin-Heidelberg-New York, 1984.
- [2] P. N. Dowling. Isometric copies of c_0 and ℓ^∞ in duals of Banach spaces. *J. Math. Anal. Appl.*, 244:223–227, 2000.
- [3] P. N. Dowling, W. B. Johnson, C. J. Lennard, and B. Turett. The optimality of James's distortion theorems. *Proc. Amer. Math. Soc.*, 125:167–174, 1997.
- [4] P. Harmand, D. Werner, and W. Werner. *M-ideals in Banach Spaces and Banach Algebras*. Lecture Notes in Mathematics 1547. Springer, 1993.
- [5] W. B. Johnson and J. Lindenstrauss. *Handbook of the Geometry of Banach Spaces, Volumes 1 and 2*. North Holland, 2001, 2003.
- [6] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces I and II*. Springer, Berlin-Heidelberg-New York, 1977, 1979.
- [7] H. Pfitzner. L-summands in their biduals have Pełczyński's property (V^*) . *Studia Math.*, 104:91–98, 1993.
- [8] H. Pfitzner. A note on asymptotically isometric copies of l^1 and c_0 . *Proc. Amer. Math. Soc.*, 129:1367–1373, 2001.
- [9] H. Pfitzner. Separable L-embedded Banach spaces are unique preduals. *Bull. Lond. Math. Soc.*, 39:1039–1044, 2007.
- [10] S. Simons. On the Dunford-Pettis Property and Banach spaces that contain c_0 . *Math. Ann.*, 216:225 – 231, 1975.

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